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BACHELOR THESIS

Scrutinizing Parametric Value-at-Risk Measure under Real-World Assumptions

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Declaration of Authorship

The author hereby declares that he compiled this thesis independently, using only the listed resources and literature.

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Signature

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Abstract

The thesis compares an industry-standard parametric Value-at-Risk estimate with alternative approaches. The intention of the thesis is to find out, whether, or to what extent can the inappropriate assumption of normally distributed returns influence the Value-at-Risk estimate. We used the exceedance rate as a back-testing procedure in order to test the accuracy of parametric Value-at-Risk estimate. We study the exceedance rate of the estimates and its difference from the theoretical value. We contrasted the parametric measure to its historical and Monte Carlo counterparts. The latter assumes Student's t-distribution as an example of a fat-tailed distribution, because the estimation of tails is crucial for the accuracy of Value-at-Risk estimate.

JEL Classification C15, G17, G32
Keywords Value-at-Risk, Value-at-Risk modeling, risk management, Monte Carlo simulation, fat tails

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Abstrakt

Tato práce srovnává v praxi často používaný parametrický odhad hodnoty v riziku s alternativními metodami jejího odhadu. Předmětem práce je zjistit, zda, a případně do jaké míry ovlivňuje předpoklad normality dat tento odhad. Pro testování přesnosti odhadu jsme použili metodu zpětného testování, která zkoumá míru, s jakou za určitý časový úsek přesahovala denní ztráta portfolia odhadovanou hodnotu, a její odchylku od teoretické hodnoty. Jako alternativní metody pro srovnání jsou použity historická metoda a Monte Carlo simulace. Metoda Monte Carlo předpokládá studentovo rozdělení jako příklad rozdělení s těžkými chvosty. Toto rozdělení lépe odhaduje rozdělení ztrát a zisků ve chvostech, jejichž odhad ovlivňuje správnost odhadu hodnoty v riziku.

Klasifikace JEL C15, G17, G32
Klíčová slova hodnota v riziku, modelování hodnoty v riziku, rizikový management, simulace Monte Carlo, těžké chvosty

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Acronyms

VaR Value-at-Risk

pVaR Parametric Value-at-Risk

hVaR Historical Value-at-Risk

VaR-x Parametric Value-at-Risk assuming Student's t-distribution with degrees of freedom equal to the tail index

VaR-t Parametric Value-at-Risk assuming Student's t-distribution with degrees of freedom equal to maximum likelihood

PDF Probability density function

CDF Cumulative distribution function

Bachelor Thesis Proposal

Author	Zuzana Rusá
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Proposed topic	Scrutinizing Parametric Value-at-Risk Measure under Real-World Assumptions

Topic characteristics The aim of this thesis is to assess the performance of an industry-standard parametric Value-at-Risk (VaR) measure vis-á-vis its simulation-based and historical counterparts, in a non-Gaussian world of financial markets. Namely, I would like to quantify the impact of (a) relaxing of some underlying assumptions and (b) data quality on parametric VaR estimates, at different time horizons.

Methodology To analyze the accuracy of parametric VaR estimates, I will contrast it to alternative estimation methods in an experimental environment. This "back-testing" procedure will help me to quantify the level of imprecision using a variety of benchmark portfolios. The benchmark portfolios will be constructed such that they, by intention, lack some of desired assumptions for the parametric VaR model. For the empirical analysis, I will use (a) last 20-year performance of major stock indices (FTSE, NYSE, Dow Jones, NIKKEI, S&P 500 ...) and (b) simulated data.

Outline

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4. Empirical Analysis
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6. Conclusion

Core bibliography

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Chapter 1

Introduction

Value at risk (VaR) is a widely used risk measure, which was first introduced in 1993. It gained its popularity due to ease of its implementation and relatively simple estimation framework, allowing also non-statisticians to use it. In order to interpret the VaR correctly, one should understand the intuition behind the estimation, including its strengths and weaknesses. We recognize three, in theory, equivalent approaches. More concretely, VaR can be estimated using (a) historical, (b) parametric or (c) simulation-based approach. Parametric and simulation-based VaRs are calculated based on some assumption concerning the distribution of returns of a portfolio, while the historical approach is distribution assumption free. Despite obvious violation of normality assumption in the financial data, parametric VaR is often used as a measure of the minimal equity capital. The question is, if and to what extent does the parametric VaR underestimate the true risk of the considered portfolio.

The aim of the thesis is to examine the performance of widely used parametric VaR compared to its simulation based and historical counterparts. By relaxing the normality assumption, we try to underpin the existing risk with the use of Student's t-distribution, internalizing the conspicuous fat-tails. The intention is to compare the aforementioned approaches (a) in the real world of financial markets, as well as (b) on simulated data. The portfolio used for comparison consists of major stock market indices. To analyze the accuracy of parametric VaR estimate, we monitor the overall performance of the method over a long time horizon. We show that parametric VaR underestimates the true exposure to risk and the deviations from the true value increase with the confidence level.

The thesis is structured as follows: Chapter 2 gives the theoretical background behind the VaR. Section 2.1 reviews the related literature. Section 2.2 briefly describes the data used for the analysis and section 2.3 presents the methodology of VaR. The last section is further divided into two parts. The first part is devoted to the three basic VaR approaches, while the second implements the Student's t-distribution in the VaR methodology. Chapter 3 is devoted to the empirical study. Section 3.1 considers the VaR of a single asset. First, it gives an insight into the statistical properties of the normal and Student's t-distribution that are important for VaR theory. Second, it presents the VaR analysis on a single asset and compares the results. Section 3.2 analyzes the VaR in a bivariate dimension. First, it touches upon the copula theory that is essential for Monte Carlo simulation. Then, it analyzes the VaR of a bivariate portfolio consisting of stock market indices. The results of the empirical study are described in section 3.3.

Chapter 2

Theoretical Framework

2.1 Literature Review

Value-at-Risk (VaR) is a risk measure developed in 1993[1], first implemented in the J.P. Morgan's RiskMetrics (Krause 2003). After the method was introduced, a lot of studies have been published covering the topic and/or proposing alternative approaches. There are three widely used methods, namely (a) historical VaR (hVaR), (b) parametric VaR (pVaR) and (c) Monte Carlo VaR. Although the method is very popular in practice, it has some considerable weaknesses, which should be acknowledged by both academics and practitioners. Krause (2003) focuses at the theory of VaR and emphasizes these limitations. On the other hand, Berkowitz & O'Brien (2002) focus on the empirical VaR results by analyzing portfolios of large banks. The inaccuracy of parametric approach motivates Huisman *et al.* (1998) to apply the theory of fat tailed distributions to VaR. Furthermore, Xu & Chen (2012) extend the univariate method to a multivariate dimension. This section provides a brief summary of the articles that have just been mentioned.

Due to the fact that VaR is widely used even among non-specialists, it is important to understand its weaknesses in order to prevent misinterpretation or overreliance on the measure. Krause (2003) provides a brief summary of what should we beware of when computing VaR. The main problem concerns the estimation of the cutoff point, which is the quantile of the future distribution of returns. The difficulties arising from the estimation the cutoff point are described in this article. First, when estimating the cutoff point, the method remains silent abut the magnitude of losses below it. The ignoring of the dis-

[1]Value-at-Risk measure is widely used since 1994

tribution below the cutoff point can result in the same VaR for two assets with virtually different risks. Second, we should keep in mind that we evaluate the historical data. More precisely the distribution of returns changes over time, namely the volatility and correlation between assets. We usually try to use the most relevant data for the estimation of future returns, so we use the most up-to-date data set. We suppose that this data set will be the most similar to actual distribution. Unfortunately, in general, this statement is not valid and we can observe serious deviations while using last year's observations. Furthermore, the intention to use small samples usually results in the lack of extreme observations in the empirical distribution. Therefore, VaR is the estimate under normal market conditions. Estimation errors are large due to the lack of tail observations and these errors increase with higher confidence level. Fitting the empirical distribution to the normal distribution eliminates the error attributed to the lack of extreme losses, but it produces errors stemming from misspecification. Krause (2003) deals with some weaknesses of VaR which cause the inaccuracy of the measure. We cannot cover all of them in the thesis, but this article gave us general insight into the problematic of VaR.

Berkowitz & O'Brien (2002) present the first empirical study of large portfolios of commercial banks. They analyze returns of six major banks at 99% confidence level over a horizon of one day as set by the regulation. Their opinion is that VaR measure is quite conservative, because only one bank experienced more exceedances than expected. This conservativeness requires banks to keep greater capital coverage then is actually needed.

For the purpose of this thesis, we decided to concentrate on the effect of the assumption on normality on the VaR measure. Namely, we compared VaR results of the parametric method to a more conservative approach taking into account fat tails of the distribution of returns. Several articles are occupied with this topic.

Huisman *et al.* (1998) focus on the generalization of Student's t-distribution into the theory of a single asset VaR. This parametric method is called VaR-x. Motivation of the study is to provide a measure which captures the fat tails of an instrument better than the parametric VaR. There exists a difference between the empirical distribution and the one estimated by normal distribution. The difference between quantiles of these distributions increases with increasing confidence level. The ssumption that returns have normal distribution does not capture the fat tails of an actual distribution, leading to an underestimation of the exposure to market risk. The study emphasizes that

VaR should be sensitive to the tails of the distribution of returns. Unlike pVaR, VaR-x provides such measure, not only does it capture the fat tails better than the parametric estimation, but it also does not assume that the distribution of returns is symmetric. The number of degrees of freedom is estimated by Hill estimator, which equals to the tail index of the bottom tail. This leads to a better estimation of tails and therefore we get a more accurate VaR measure at higher confidence levels. The study is concerned with single asset VaR at different confidence levels. Its empirical part shows that VaR-x only give more conservative estimates for high levels of confidence starting around 97%. For a 95% confidence level, the pVaR estimate is, by construction of the VaR-x, more conservative. There are several studies following Huisman *et al.* (1998) in which other alternatives to pVaR are proposed.

Lin & Shen (2006) perform an empirical study comparing three approaches. Except the pVaR and VaR-x, they analyse the performance of VaR-t, which is also a parametric approach assuming returns with Student's t-distribution. Unlike the VaR-x, this approach does not set degrees of freedom equal to the tail index; it estimates the degrees of freedom by estimating the excess kurtosis of the whole sample. They found out that the VaR estimation using Student's t-distribution can improve VaR estimate and provide more precise results. Similar results can be found in Rozga & Arneric (2009). All of these studies deal with univariate VaR, the extension to multivariate case provide Xu & Chen (2012).

They evaluate several techniques of VaR on a bivariate portfolio. Two approaches are considerable for our purpose to provide an estimate of VaR using Student's t-distribution. Namely it is a multivariate t-distribution model and Copula-based Monte Carlo approach. They showed that the simulation method gives more accurate results; therefore we will use this method in contrast to the parametric approach.

To sum up briefly, the normality assumption is a weak point of the popular parametric method. It is questionable, to what extent this measure may underestimate the true VaR, so we will contrast the parametric method to alternative approaches, which does not assume normally distributed returns.

2.2 Data

The data used for the thesis were stock market indices, namely S&P 500, FTSE 100 and DAX from EODData database [2] and analyzed in Matlab[3]. This historical data set consists of 5803 daily closing stock prices covering the period between January 1, 1993 and March 31, 2015.

2.3 Methodology

In this section we describe common techniques used to compute VaR estimates of a portfolio, which are later applied in the empirical study. First, we look at what the VaR stands for. Choudhry (2013) defines value at risk as the maximum loss that can occur with given confidence level over a target horizon. The purpose of VaR is to measure the market risk. According to Jorion (2007), the loss consists of the volatility of the financial variable and the exposure to this particular risk. VaR captures combined effect of both. VaR should satisfy the equation

$$\mathsf{P}\left(L > \mathsf{VaR}\right) \leq 1 - \alpha,$$

where α is the confidence level (for example 95%) and $L = -W_0 R^{\star}$ represents the portfolio loss as a positive number. Defining $f(w)$ as the probability distribution of the future portfolio profit, we are looking for $R^{\star}$ such that

$$1 - \alpha = \int_{-\infty}^{R^{\star}} f(r)dr, \tag{2.1}$$

with given α. An important assumption concerning all the mentioned VaR approaches is that the portfolio does not change over the target horizon. Culp *et al.* (1998) stress that this assumption may be problematic while computing VaR for a long time horizons. Although this assumption may not be fulfilled in practice, addressing the implications is not the scope of this thesis. Indeed, the intention of this thesis is to study further the inaccuracy of the parametric VaR resulting from the violation of the normality assumption. Next subsection gives us an insight into the techniques of VaR estimation approaches.

[2]All data were downloaded on April 1 2015 from the EODData database - `www.eoddata.com`

[3]Matlab, version 2014b, was used to perform all the simulations. Codes are available at request.

2.3.1 VaR Methodologies

According to Russon (2008), there are three basic approaches for the VaR estimation, namely the (a) historical method, (b)parametric method and (c) Monte Carlo method. He also argues that these methods should converge. In other words, any difference in the respective VaR estimates is caused by modelling issues and/or violation of assumptions. Independently of the method used, there are four steps that should be completed according to Choudhry (2013). Firstly, it is important to choose the time horizon over which we want to compute VaR. It should reflect the time needed for portfolio liquidation. It also depends on the data that are available (Russon 2008). 1, 10 or 20 day time horizons are commonly used to measure the loss. The second step is to select the level of confidence, usually a 95% or 99% level of confidence is chosen. In the third step, we create a probability distribution of future returns of a portfolio. This step can be divided into two parts. Culp *et al.* (1998) suggest to estimate distributions of individual instruments and then model the distribution of overall portfolio using marginal distributions and appropriate correlation measure. Individual approaches have different assumptions about the future distribution, being a major cause for differences between their respective estimates. The last step is to compute a VaR estimate given the previous steps.

Unlike the distribution of future returns, the choice of quantitative factors depends on the use of the VaR estimation rather than the method applied for its computation. The longer the time horizon or the greater the chosen confidence level, the bigger VaR we obtain. Jorion (2007) introduces three frequent applications and corresponding factors used in practice. First, if VaR is used as a benchmark measure, it is more important to stay consistent in choosing the quantitative factors than its size. Banking industry uses a 99% confidence level and time horizon is equal to one day in order to follow the Basel Accord. The second common application of VaR is to measure a potential loss of a portfolio. For that purpose, the risk horizon should reflect the liquidation period of the portfolio. The third application of the VaR is to determine equity capital. In this case, we have to choose factors carefully, because a loss greater than VaR can lead to bankruptcy.

Before we look at the methods used in the empirical part, it is essential to define returns. There exist two ways how to express return of a portfolio or an asset. First way to define returns is the rate of return. Butler (1999) expressed return as:

$$R_{t+1} = \frac{P_{t+1} - P_t}{P_t}. \tag{2.2}$$

The second way, how to define returns is a logarithmic form (e.g. Choudhry (2013))

$$r_{t+1} = \log\left(\frac{P_{t+1}}{P_t}\right), \tag{2.3}$$

Where P_t is yesterdays' price of an instrument or a portfolio and P_{t+1} is today's price. In the empirical analysis, we used the second type for computations, but the results are expressed in the rate of return, being easier to interpret.

According to Jorion (2007), VaR can be expressed as an absolute or relative dollar loss. Relative VaR is the dollar loss relative to the mean while the absolute VaR express the dollar loss relative to zero. Both expressions lead to similar results if the horizon is short (mean of the return is around zero). The formula for absolute VaR, which is applied in the empirical part, is following:

$$\mathsf{VaR}(zero) = -W_0 R^\star, \tag{2.4}$$

where W_0 is the initial value of a portfolio and $R^\star$ is the rate of return corresponding to the loss at given confidence level.

In the subsequent part of the thesis, we will describe in detail individual VaR approaches. First, we look at the historical method, which uses the historical distribution of returns to predict the future one. Second, the parametric VaR estimation is explained. This method uses the historical returns to estimate parameters of a normal distribution, which then determine the future distribution of returns. Finally, a Monte Carlo method, which assumes that future returns follow a known stochastic process, is described.

Historical VaR

Historical VaR method is the simplest among the three VaR methodologies. According to Russon (2008), this method is useful for securities with long and liquid history. Cheung & Powell (2012) contrast the historical VaR to the parametric measure by saying that unlike the parametric method, historical VaR avoids misspecification of the future distribution. This misspecification caused by the violation of the normality assumption can lead to the over or underestimation of the true VaR. Historical approach assumes that the future distribution of returns is identical to the historical one. Therefore, the accuracy of historical estimation relies on whether the history indicates the future well.

Cheung & Powell (2012) also discuss the choice of the size of the historical sample that is used to illustrate the future distribution. Their opinion is that the longer the sample is, the more information it contains and therefore is preferred. On the other hand, Hendricks (1996) points out the advantage of short sample. Short sample captures the recent movement of the portfolio. On top of that, Russon (2008) thinks, that the sample size should be long enough to contain at least 5 observations smaller than the VaR estimate.

Having the appropriate sample size, we compute hVaR by multiplying the $(1-\alpha)\%$ quantile of the empirical distribution by the initial value of the portfolio. This approach is difficult to implement on a complex portfolio that changes over time.

Parametric VaR

Parametric VaR estimate (pVaR) imposes an assumption of specific distribution. The existence of a closed-form formula is a characteristic of pVaR. Normal distribution is often used as a benchmark distribution. Following part describes the parametric method assuming normality. The theory assumes that the returns follow multivariate normal distribution with a constant mean and variance. According to Russon (2008), there are a few steps we have to follow in order to get to pVaR. First, we estimate the mean and standard deviation for each asset in the portfolio, then we estimate a correlation matrix of the portfolio and finally we use the formula to calculate pVaR. To estimate all parameters, we choose the same data set as in historical method, in order to obtain comparable results.

A formula for calculation of a single asset pVaR at time t defined by Rozga & Arneric (2009) is following:

$$\mathsf{VaR}(\alpha) = -W_0\, z_{1-\alpha}\, \sigma, \tag{2.5}$$

where α is the confidence level, W_0 is the initial asset value and σ is estimated parameter and $z_{1-\alpha}$ is a $(1-\alpha)\%$ quantile of the standard normal distribution. By convention, the loss is expressed as a positive number. Multivariate pVaR is computed with the help of variance-covariance matrix. The formula written for example by Choudhry (2013) is following:

$$\mathsf{VaR}(\alpha) = -W_0\, z_{1-\alpha}\, \sigma_p, \tag{2.6}$$

where σ_p is the standard deviation of the portfolio. The standard deviation of the portfolio is computed using the formula:

$$\sigma_p = \sqrt{w' \Sigma w}, \tag{2.7}$$

where w represents proportional weights of individual assets in a portfolio and Σ stands for the covariance matrix.

We described the intuition behind the parametric methodology. It is a simple theoretical approach which is built on the normality assumption, even though returns rarely approach normal distribution as written by Huisman *et al.* (1998). The alternative may be Student's t-distribution, because it better represents the tails of the distribution. Parametric VaR estimate can deviate from the true value due to the violation of return assumption. To what extent we can suppose it happens is further studied in the empirical part. We also study the difference between pVaR and an alternative approach assuming Student's t-distribution.

Monte Carlo VaR

Monte Carlo method is the most flexible method from the three mentioned. Similar to parametric approaches, this approach also assumes that the distribution of returns follow a stochastic process. Unlike parametric approaches, the desired metrics are derived from the Monte Carlo simulation, rather than a closed-form formula. For more details on Monte Carlo simulation, see Jorion (2007). Choudhry (2013) claims that the more simulations we run, the more accurate VaR estimation we get. Random numbers are usually drawn from a normal or log-normal distribution, and the stochastic process is parametrized by standard deviations and correlations. These parameters are usually estimated from a historical data set. Cheung & Powell (2013) mention, that the stochastic process for the price of a share has the form of geometric Brownian motion.

$$S_{t+\Delta t} = S_t \,\exp\left(k\,\Delta t + \sigma \epsilon_t \sqrt{\Delta t}\right), \tag{2.8}$$

where S_t is the price of a share, Δt is the time horizon expressed as the portion of a year, k is the drift ($k = \mu - \sigma^2/2$) and ϵ_t is a random shock generated from given distribution. By an easy transformation, we obtain the distribution of returns, which is Gaussian. Russon (2008) proposes two ways of computing VaR out of the simulated distribution. One of them is based on the historical approach by the use of the empirical quantile. The other one performs the parametric method on the simulated distribution of returns.

2.3.2 Student's t-distribution and VaR

Recalling the Huisman *et al.* (1998) article, the return distribution of an asset rarely follows a normal distribution. Furthermore, we observe that the distribution has fat tails, which means that the empirical values fall under the tail more often than would be predicted by the normal distribution. Therefore they introduce a measure called VaR-x, which assumes a fat tailed distribution, namely Student's t-distribution, in order to get a better fit in the left tail of the distribution of returns. Correspondingly, we suppose, that the better fit will produce more accurate VaR estimates. Huisman *et al.* (1998) only consider a single asset VaR. The methodologies used later in the empirical part differ. The univariate VaR is computed by a parametric method while the multivariate VaR estimation uses Monte Carlo simulation. The common feature is the assumption, that the distributions of future returns follow Student's t-distribution.

Concerning the univariate case, Huisman *et al.* (1998) and also Rozga & Arneric (2009) use a parametric model to estimate VaR assuming Student's t-distribution. The degrees of freedom are equal to the left tail index of the distribution, e.g. the Hill estimator. The advantage of this particular extreme value theory approach among others is that it only requires small data sample. Let k be the number of left tail observations and x_i be sorted absolute values of these observations in ascending order. We can estimate the degrees of freedom by simply inverting the estimator of the tail index α. We compute the following:

$$\gamma(k) = \frac{1}{k}\sum_{j=1}^{k} \log(x_{n-j+1}) - \log(x_{n-k}), \tag{2.9}$$

where n is the number of observations and $k = 1, \dots, \lfloor n/2 \rfloor$. Next we fit a simple linear regression model for the outcome $\gamma(k)$ with a covariate $k, k = 1, \dots, \lfloor n/2 \rfloor$. The inverse of the intercept estimates the tail index α, so it is possible to estimate the degrees of freedom (Huisman *et al.* 1998). Having estimated all the parameters, we can rewrite the parametric formula expressed in equation 6 to get the parametric formula for VaR-x:

$$\mathsf{VaR} - x(\alpha) = -W_0\, t_{1-\alpha}^{df}\, \sigma \sqrt{\frac{df-2}{df}}, \tag{2.10}$$

where $t_{1-\alpha}^{df}$ represents the critical value of Student's t-distribution with estimated degrees of freedom given confidence level α. The square root at the end of the formula represents the correction factor, which is included in the formula in order to obtain an unbiased estimator of the standard deviation. As it was already said, this measure provides more conservative results than pVaR at high confidence levels, but it is not valid for lower levels such as 95%.

In the section assuming normal distribution of the returns, we generalised the univariate case to the multivariate case simply by the use of the multivariate normal distribution. Starting with the same idea, we could try to implement the method using a multivariate t-distribution with a specific mean and covariance matrix. This method is described in Xu & Chen (2012) where he states that the performance of this technique is poor compared to another one which we will pursue further.

The method we use to compute multivariate VaR measure using Student's t-distribution is a copula-based Monte Carlo approach. Monte Carlo simulation requires the estimated joint distribution of future returns. This method is sensitive to the assumptions concerning the joint distribution of a portfolio; therefore the deviation from the true future distribution may lead to inaccuracy. Copula is a statistical tool developed to generate joint distribution regardless of the marginal distributions. For the thesis, we choose the industry standard Gaussian copula. More information on copulas can be found in Jaworski *et al.* (2010).

Xu & Chen (2012) described the procedure of implementing copulas into a Monte Carlo simulation method in the following way:

1. We choose a copula family and marginal distributions of individual asset returns. We also have to estimate the parameters of the marginal distributions. In this thesis, we use a Gaussian copula with marginal t-distributions.

2. Using the copula and marginal distributions, we estimate the parameters of the joint distribution.

3. We simulate enough random variables from the joint probability density, then by the use of the standard normal CDF we transform the random variables. Finally, we invert these variables by the use of the marginal CDFs.

4. We calculate the portfolio returns and estimate the empirical VaR.

This method is quite complicated and the computing time is long. We will focus on the performance of this method compared to the multivariate pVaR model. In the empirical study, we focus on the performance of pVaR and we study how much deviation from the true VaR can be caused by the violation of the normality assumption.

Chapter 3

Simulation and application

Daily closing prices of major stock market indices, namely S&P 500, FTSE 100 and DAX, are used to analyze the VaR estimates. The aim of this study is to look at the performance of an industry-standard parametric Value-at-Risk measure by contrasting it to alternative approaches. The structure of the empirical study is as follows: The chapter is divided into two sections. Section 3.1 is concerned with univariate VaR. This simple study is sufficient to give us a basic idea of weaknesses we can expect from parametric VaR estimation.

Section 3.2 deals with a bivariate case, which resembles the reality as we are usually interested in VaR of a portfolio and not a single asset. Methods and techniques applied in this section are easily extendable to a multivariate level. Both sections are further divided into two parts; that is a simulation and an empirical study. Simulation study illustrates the possible weaknesses of parametric VaR arising from the violation of assumptions. In the empirical part, we test whether these worries are fulfilled in the real world of stock market or not.

3.1 Univariate VaR

The parametric Value-at-Risk (pVaR) is a widely used measure mainly for its computational simplicity. Parametric VaR heavily relies on the assumption on the return distribution. The industry standard pVaR implementations typically assume normal distribution of returns, even though this is rarely the case in the real world. The actual distribution of returns is skewed and leptokurtic. As we are interested mainly in the tails of the distribution, the normality assumption can cause serious disparities between the measured and real exposure to risk.

Indeed, it is the goal of this thesis to quantify the potential losses arising from these disparities. Looking at the distributions of different assets, we can see that most of them tend to have fat tails. It means that the empirical distribution hits low (potentially high) values more often than predicted by the normal distribution. As a result, we decided to estimate returns by a distribution which would better fit the tails of the empirical data.

3.1.1 Simulation study

For the purpose of this study, the Student's t-distribution is used as an example of a fat-tailed distribution. Figure 3.1 shows the Student's t-distribution's PDF with different degrees of freedom compared to the standard normal PDF. As the number of degrees of freedom increases, t-distribution converges to the normal distribution (i.e. the lower number of degrees of freedom, the fatter the tails). Figure 3.1 also compares quantiles of Student's t-distribution with low degrees of freedom with the quantile of a normal distribution. The vertical lines represent the five percent quantiles of corresponding distributions. A five percent quantile of normal distribution is equal to -1.6449. Assuming Student's t-distribution with 5 degrees of freedom, the value drops to -2.0150 and for 3 degrees of freedom, the quantile decreases to - 2.3534. For illustration, if we invested $1,000,000 in an asset with the estimated volatility of one percent, there would be a five percent chance that the loss will exceed $16,449, $20,150 and $23,536 respectively in one day. The impact severity arising from degree of freedom estimation is even greater considering one percent VaR. It is evident that the choice of the distribution is crucial for the parametric VaR estimation.

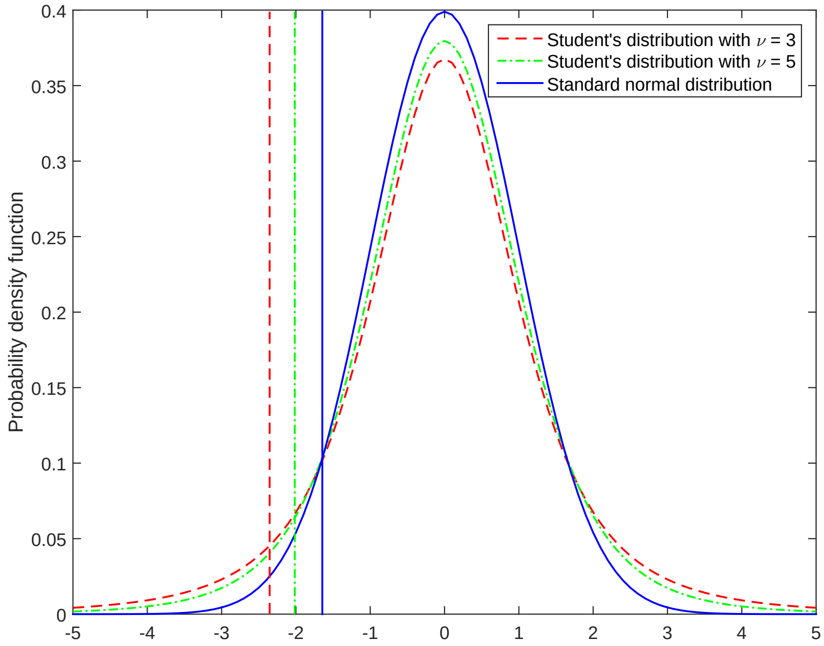


Figure 3.1: 5% quantile of Student's distribution.

The estimation uncertainty is yet another important feature in VaR estimation. It is a consequence of estimating parameters from a sample instead of the whole population. This feature is mainly concerned by practitioners and empirically minded academics. We look at the estimations of the one percent quantile of a standard normal distribution coming from different sample sizes. We compute an empirical quantile of the simulated vectors and compare it to the true value, which is -2.3263. The sample sizes are the following: 125, 250, 500, 1,000, 5,000, 10,000 and 100,000. For each sample size we run a Monte Carlo simulation to obtain VaR estimates. We run the Monte Carlo simulation 500 times and we look at the distribution of estimators of one percent quantile. Generally, the parametric VaR estimate converges to the true VaR as the sample size increases, as depicted at Figure 3.2. It implies that the more observations we have, the more accurate VaR measure is. On the other hand, it does not mean that historical VaR is more accurate when we compute the quantile using more historical observations. It is due to the fact that historical distribution does not necessarily demonstrate future distribution of returns well. The historical method is not the main interest of this thesis so we thereby leave the discussion aside. To sum up briefly, the intention of this work is to illustrate how inaccurate results we can get using the simple parametric method in the market environment that is not normal.

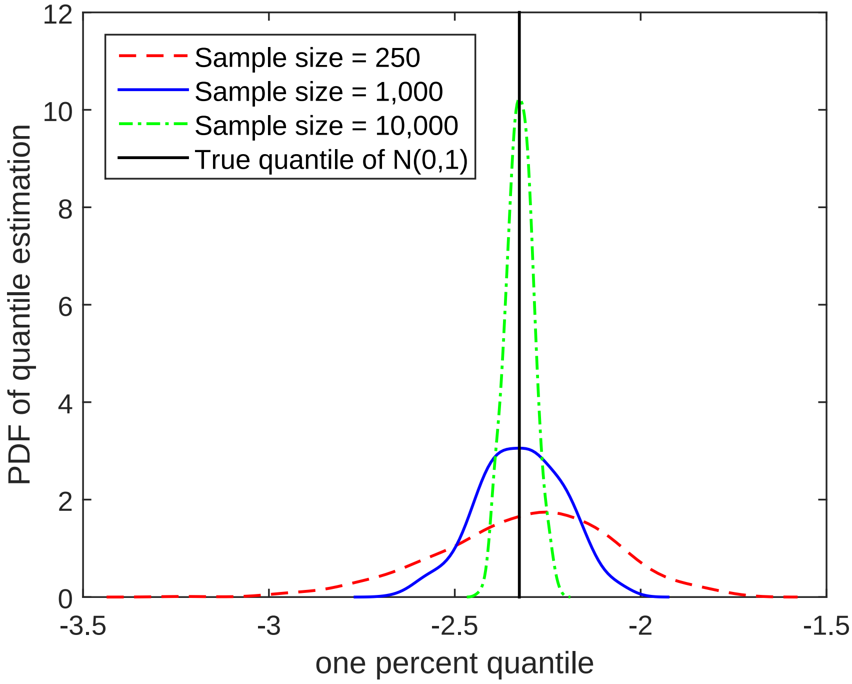


Figure 3.2: Estimation uncertainty.

3.1.2 Empirical study

This part focuses on a VaR of a single asset. Three VaR estimation approaches are compared using a 500 day historical sample. Parametric VaR measure, using both normal and student return distribution assumption, is contrasted to the historical VaR (hVaR)estimation. At time t, historical sample of last 500 daily returns is used to compute 1-day VaR at a given confidence level α ($\alpha = 99\%$). The number of returns in the sample size is set such that there are at least 5 observations within the one percent quantile. It was proposed by Krause (2003).

Historical VaR is a one percent quantile of the historical distribution of the empirical stock index returns. To estimate the normal pVaR, sample mean and variance were fitted to the historical returns, see section 2.3.1. These parameters are as well estimated from the sample size of 500 historical returns. It is due to obtaining comparable results. To estimate VaR-x using formula from equation 2.10, we need to estimate one more parameter. The number of degrees of freedom is estimated by the Hill estimator(see section 2.3.2) for every rolling window. The estimated degrees of freedom are between three and seven during the analysis lasting 500 days.

Results of applied methods are shown in figure 3.3; The blue solid line represents the returns of particular stock market index, the red dashed line pVaR, the green dash-dotted line hVaR estimate and the black solid line states for VaR estimate using Student's t-distribution (VaR-x). Most of the time, hVaR is the most conservative out of the three compared methods. On the other hand, pVaR is above all which indicates that the number of exceedances will be bigger than in other two measures. For the S&P 500, the model experienced 3 exceedances of hVaR, 12 of pVaR and 8 of VaR-x. In case of the FTSE 100 index, 6 exceedances of hVaR, 9 of pVaR and 9 of VaR-x were observed. Actual loss of DAX index was greater than hVaR, pVaR and VaR-x 6, 9 and 6 times respectively.

Recall that the confidence level was set to 99%. By construction, we expect 5 exceedances within a two year interval (500 days). We can say that none of the methods satisfies this statement for all assets. It is given by estimation error and violation of assumptions. For a historical method, the reason is the difference between the distribution of returns in the future than in the past. In case of other two measurements, the distribution does not exactly fit to the one which is used to predict future returns. Despite more exceedances than expected, it is clear that more fat-tailed distribution, such as Student's, is more conservative than normal one and gives more conservative results.

On the contrary, being more conservative does not mean being better. Figure 3.4 provides an argument to support this opinion. It shows the normalized empirical distribution fitting to the normal distribution and t-distribution with 5 degrees of freedom. Normal density better represents the density around the mean of stock returns, but it underestimates the empirical distribution in tails so it is probably not the best estimation for the purpose of measuring VaR. The thesis deals with these methods later on so there is a time to examine their performance in a more complex environment.

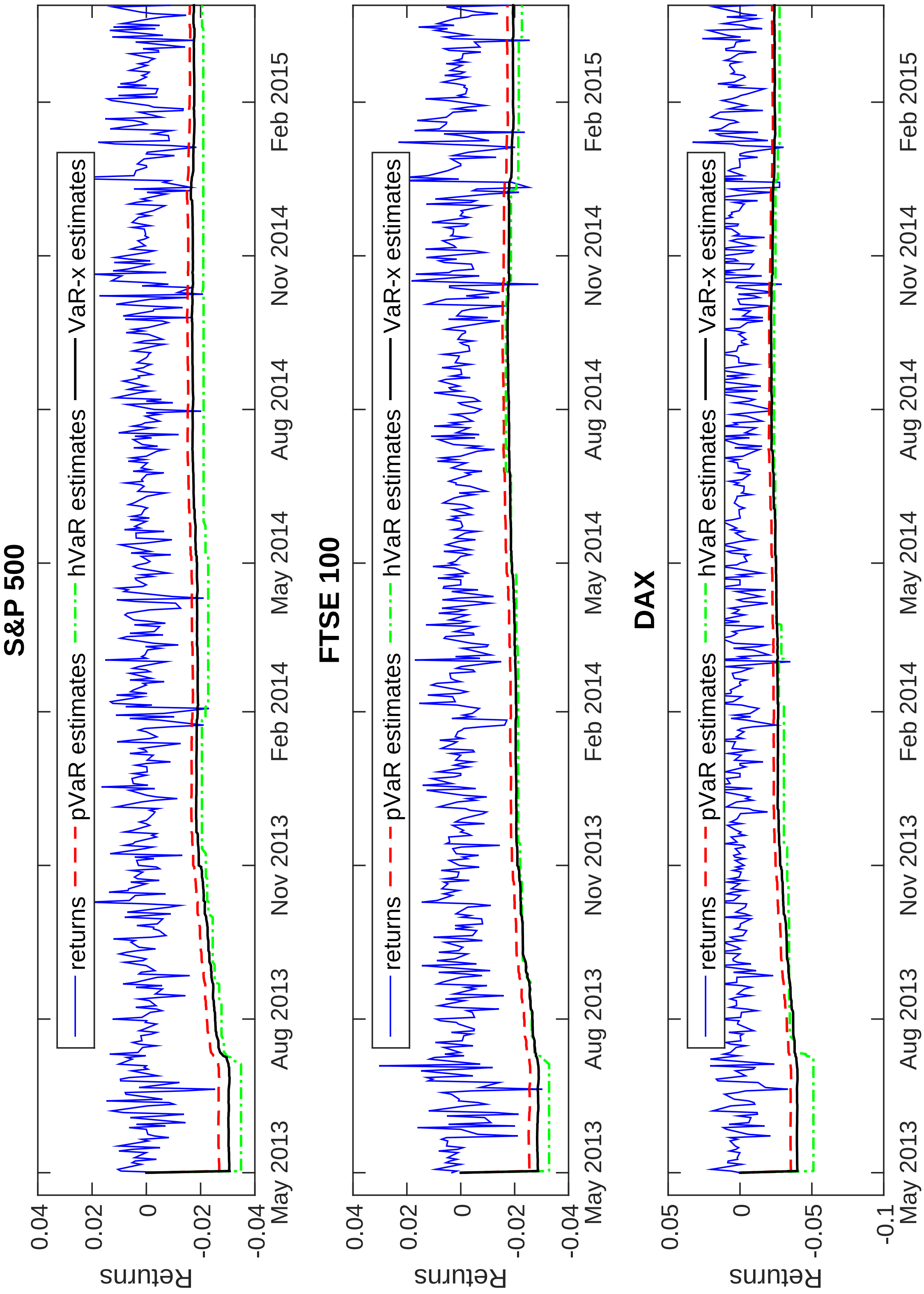


Figure 3.3: Univariate VaR of stock market indices, 99% confidence level.

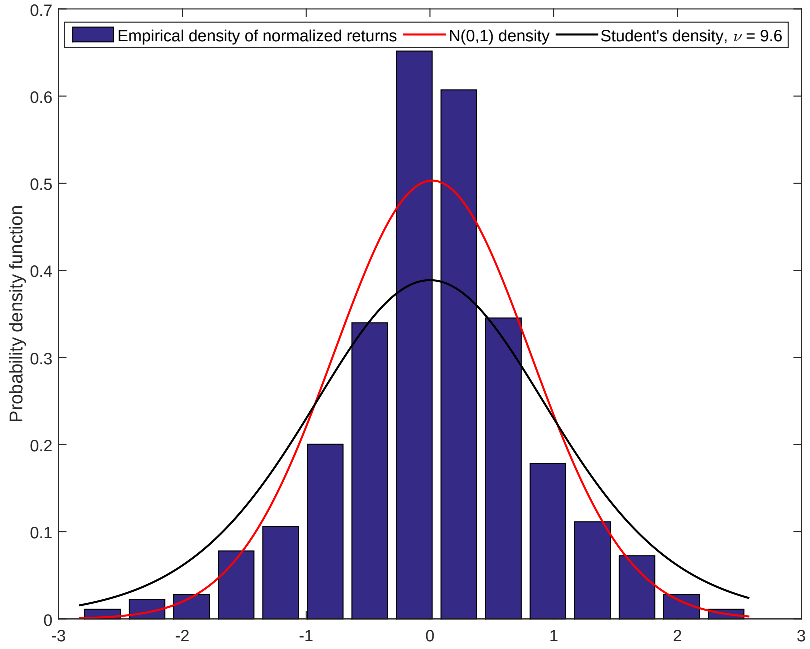


Figure 3.4: Fit of the empirical distribution.

We are also interested in the performance of the three approaches in case we compute the VaR at 95% confidence level. The results do not follow the intuition of preceding example as indicated by Huisman *et al.* (1998). Although it would be natural, if VaR computed with the use of Student's t-distribution were more conservative than the parametric one, the reality is rather counter-intuitive. For a 95% confidence level, exceedances related to S&P 500 index and a sample size of 250 preceding returns were following (the expected count was 20): 27 for hVaR, 33 for pVaR and 35 for VaR-x . VaR estimates of remaining indexes experienced similar behaviour.

The reason for that can be explained by the formula for the parametric VaR measure using Student's t-distribution, see equation 2.10. By definition, the correction factor can notably change VaR estimate. Figure 3.1 shows the placement of quantiles of normal distribution and the Student's t-distribution with $\nu = 5$. The five percent quantile of Student's t-distribution is further from the mean than the quantile of normal distribution, so it looks like the estimation should be more conservative. It is not, because of the correction factor, which is in the formula in order to guarantee unbiasedness of the estimate, reduces the difference. Figure 3.5 displays the VaR estimates for a 95% level of confidence. The parametric VaR is slightly more conservative than VaR-x.

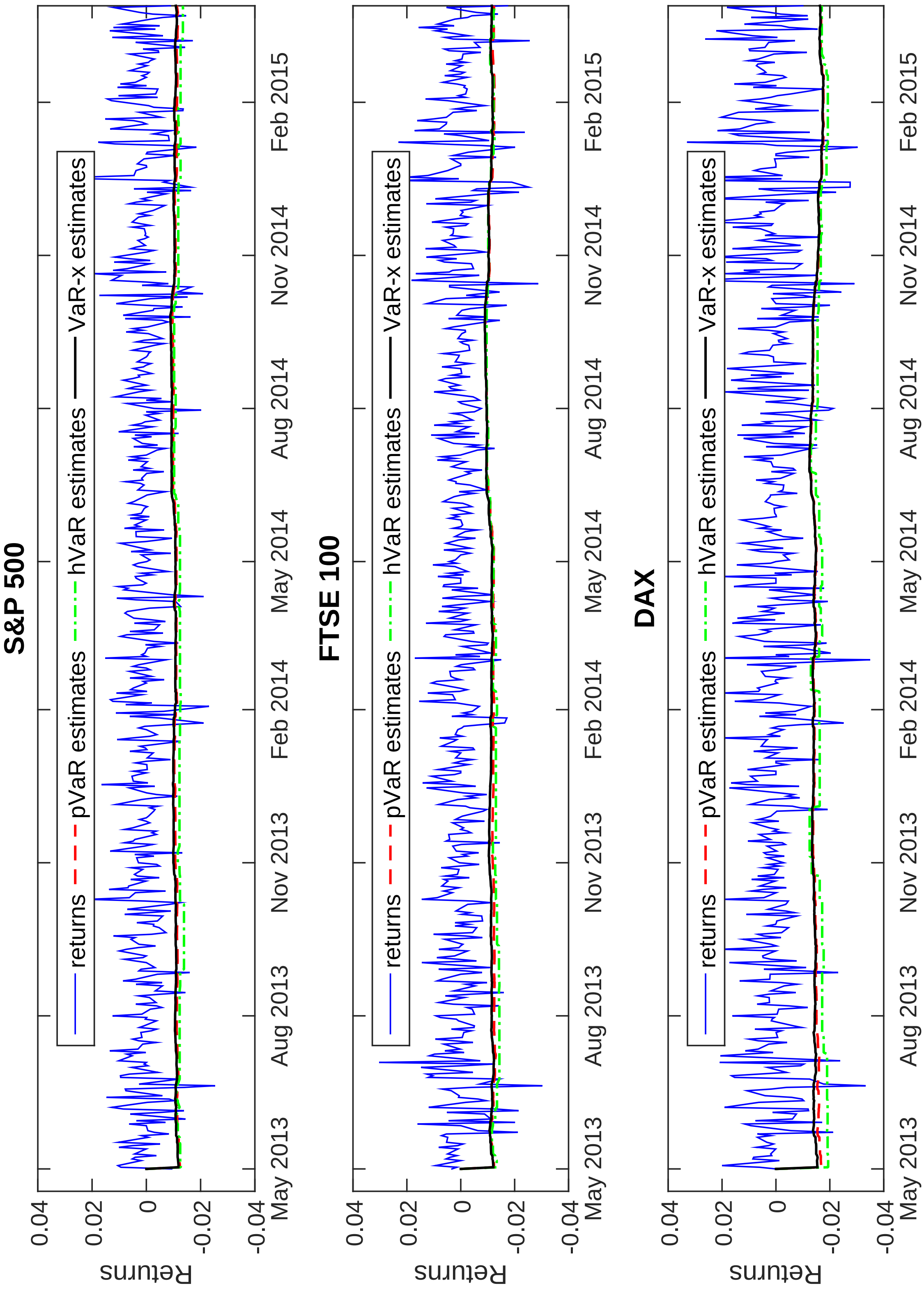


Figure 3.5: Univariate VaR of stock market indices, 95% confidence level.

Univariate study showed us that the VaR estimation is more conservative for a 99% confidence level; however, this statement is not valid for a 95% confidence level. The contradiction with the opinion of the writer was caused by the correction factor. We will see whether it works or not for a multiple asset portfolio. In the second part of the analysis, the formula for estimating VaR-x (equation 2.10) is replaced by Monte Carlo simulation. Therefore, it is early to make any conclusion.

3.2 Multivariate VaR

The second part of the empirical part is focused on the multivariate VaR. For the sake of simplicity, we focus on the bivariate case (i.e.portfolio consisting of two assets). In the simulation study, the thesis explains a method of generating vectors that have a predefined dependency structure and a given distribution (i.e. copula function). It also demonstrates how the parametric VaR depends on the assumption of underlying distribution. A copula is later used in the empirical part to simulate returns for the estimation of VaR using the Monte Carlo method. For the Monte Carlo simulation, we try Student's t-distribution as an example of fat-tailed distribution. We work with 22 years of data in order to propose the reader a well-grounded comparison of introduced methods. The data used for the analysis are described in section 2.2.

3.2.1 Simulation Study

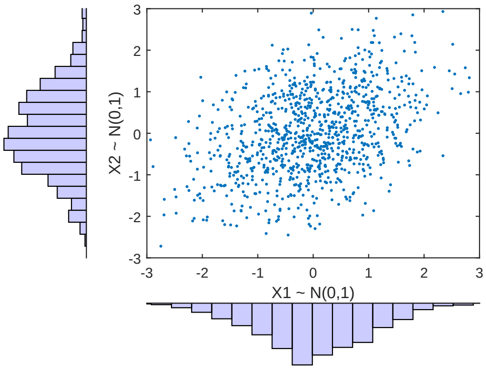
Before we move to the empirical study, it is useful to show how the assumption of normality influences the parametric VaR estimates. In practice, it is common to assume a multivariate Gaussian world, even if the true asset return distribution is far from Gaussian. The following part deals with such situation, when we compute pVaR estimate knowing that the return distribution is fat-tailed.

In this part, we simulate random vectors with a fixed correlation which follow a specific distribution. We focused on the bivariate case, therefore we need to simulate two random variables. Each of the variables may follow some predefined distribution and they have some predefined dependency structure. If the variables were normal and their dependency structure was defined by Pearson correlation coefficient, it would be the case of bivariate Gaussian distribution. While relaxing some of these assumptions (e.g. normality or Pearson correlation), the situation is more complicated and the bivariate distribution is not specific anymore, but more general. We want to relax the normality assumption and substitute it with the distribution capable to estimate fat tails. In our case, we choose the Student's t-distribution. A copula is a standard tool, which enables to separate modelling of (a) marginal distributions of random variables and (b) mutual dependence between them. This is exactly what we need for our analysis. For further information on copulas, see Jaworski *et al.* (2010). The theory of copulas is beyond the scope of the thesis, hence we only restrict ourselves to application. To be more specific, we apply the easiest type of copula - a Gaussian copula. As marginal distributions, we use the Gaussian and Student's t-distribution respectively. We found a code relevant to the application of copulas in Matlab documentation[1]. Figure 3.6 shows the copulas' output with different marginal distributions and correlation coefficient.

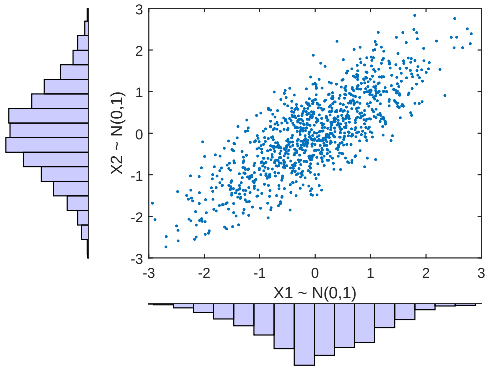
Our aim is to compare the parametric VaR for bivariate distributions under both Gaussian and non-Gaussian regime. The non-Gaussian regime is represented by the Student's t-distribution with five degrees of freedom. The formula for the parametric VaR is described in equation 2.6. The sample size of our experiment is 10,000. We will run the simulation four times for three pairs of variables. The first pair follows a bivariate normal distribution, the second consists of one normally distributed and one Student's t-distributed, the third is a copula function with marginal t-distributions. In all scenarios, both variables are equally weighted in an imaginary portfolio. We evaluate the parametric VaR measure for non-normal variables in the following paragraphs. It is a model situation, so we assume that the value of a portfolio (W_0) is 1.

In the first scenario, we calculate VaR for 99% level of confidence and a correlation coefficient equal to 0.4. Given equation 2.6 and assuming that both variables have unique variance, the parametric VaR estimate is 1.9464. Now we simulate three pairs of variables as listed above and we compute the empirical VaR. Results are following: 1.936 for the copula with normal margins, 2.4669 for the copula with one normal and one student margins and 2.812 for a copula with two student margins. There is a significant difference between the values that we obtained for the second and the third pair. This difference has two explanations. Firstly, the variables generated by copula do not have a unique variance, which is caused by dependency. Secondly, it is due to non-normal distributions. Imagine we have three portfolios, each of them contains one pair

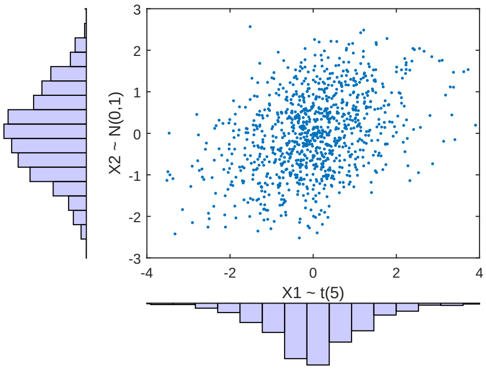
[1]Simulating dependent random variables using copulas - `http://www.mathworks.com/help/stats/examples/simulating-dependent-random-variables-using-copulas.html`



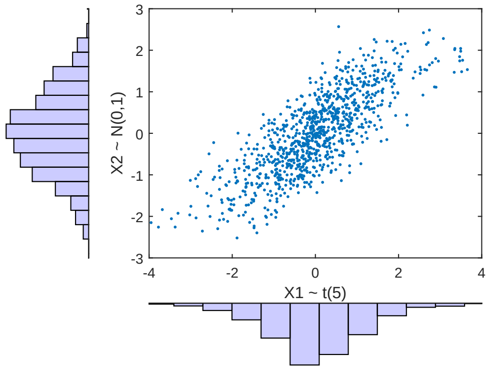
(a) Dependent Normal variables, $\rho = 0.4$.



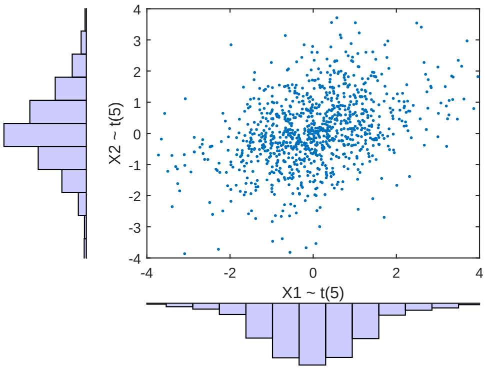
(b) Dependent Normal variables, $\rho = 0.8$.



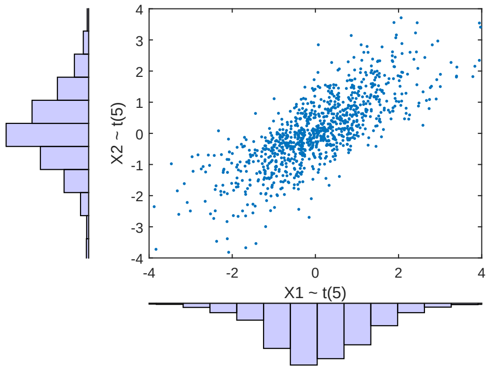
(c) Dependent Normal and t_5 variables, $\rho = 0.4$.



(d) Dependent Normal and t_5 variables, $\rho = 0.8$.



(e) Dependent t_5 variables, $\rho = 0.4$.



(f) Dependent t_5 variables, $\rho = 0.8$.

Figure 3.6: Copulas (1000 generated observations).

of variables from above (we already have the copula outcome). By applying the parametric formula (equation 2.6) to all of them, we obtain 1.9464, 2.2270 and 2.4872 respectively. The results are similar to those that we computed as a quantile of the joint empirical distributions. The empirical quantile and the pVaR estimate slightly differ for bivariate normal, but in respect to other two pairs the difference is much greater. Particularly, for the combined portfolio, the difference is 0.23 and for the portfolio consisting of two student margins, it is 0.32. In this case, pVaR underestimates true risk. In the next scenario, we will see, whether the deviation increases with higher correlation coefficient.

Hereby we set the correlation to 0.8, for each combination of marginal distributions. The expected pVaR, as calculated using the formula 2.6, is 2.2070. Empirical quantiles representing VaR are 2.2110, 2.8019 and 3.2740. The values are sorted in the same way as in the first scenario. The pVaR applied directly on generated vectors leads to the following results: 2.2218, 2.5185 and 2.8156. Compared to the results of the first scenario, the differences increased. For the copula with two student margins, the difference increased to 0.45, expressed in percentage, it is about 16%. In case of the copula with one normal and one student margins, the difference is 0.28. Based on our assumptions and results, it seems that an inappropriate assumption on the underlying return distribution leads to a significant error in VaR estimation at 99% confidence level. The higher the correlation is the greater deviation from the true loss we observe.

Finally, we look at the same study, but computing VaR at 95% confidence level. Here, the pVaR estimates are higher than the empirical VaR for both correlation coefficients. It means that parametric approach predicts greater loss than is the true one at a 95% confidence level. Unlike the case of 99% confidence level, the differences between the values are not as large. This contradictory result is caused by the shapes of joint distributions both methods assume. Consider the case of the copula with student margins. The copula has higher density around 0 than the multivariate normal distribution and it also has heavy tails compared to the fitted. Despite being more concentrated around zero, the copula also has heavier tails. However, the PDFs cross somewhere between one percent and five percent quantiles, which explains why the pVaR estimate is slightly more conservative than an empirical one at 95% level of confidence. The PDFs of both distributions with five percent quantiles are displayed in figure 3.7.

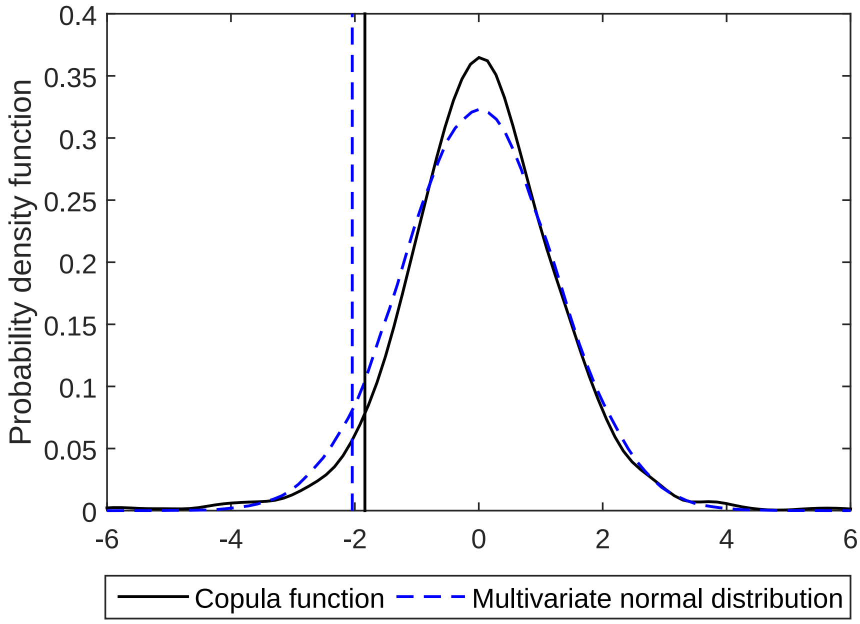


Figure 3.7: Simulated 5% quantile - Copula function and multivariate Normal distribution.

3.2.2 Empirical study

The intention of this part is to look at how accurate results parametric VaR measure assuming normal distribution performs under the real market conditions. This analysis is performed on a long time horizon in order to cover as much scenarios as the stock market can offer as possible. The method chosen to test the accuracy of parametric measure is again the exceedance rate. An exceedance occurs when the actual loss is greater than the estimated loss and it is a percentage of days when the exceedance occurred out of all days. Expected exceedances are set according to the confidence level and number of trading days included in the analysis. It is a simple back-testing technique. In case that the parametric model fits the data well, the empirical exceedance rate should be equal to the exceedance rate estimated by a parametric model.

Moreover, this widely-used parametric approach is contrasted to alternative measurement methods, that is (a) a historical value-at-risk and (b) a simulation-based method using Student's t-distribution to predict future return distributions of individual elements. The VaR exceedance rate is the only back-testing method applied in the thesis however it is not a sufficient measure to claim that one method is always better than the other one. Nevertheless, it may be sufficient for the illustration of the fact that a misspecification of return distribution assumptions can lead to serious deviations of pVaR estimates from the actual losses of our portfolio. For this simulation, a portfolio consisting of S&P 500 and DAX stock indiices was evaluated in a horizon of 22 years. Both indices are equally weighted in the portfolio. We focus only on 1-day VaR at 99% and 95% confidence levels.

Two asset portfolio model description

In this subsection, we describe the development of the implemented model. First of all, we compute the log returns of a portfolio (2.3). Then we compute the hVaR estimate for the confidence level $\alpha = 95\%$ and $\alpha = 99\%$ that we have chosen - the most common values used in practice. For a 95% confidence level at time t, we use 250 preceding returns to compute empirical five percent quantile. This quantile predicts the worst loss that a portfolio may experience at time t at a 95% level of confidence. By convention, this loss is expressed as a positive number. The portfolio return at time t is known, so we look whether the actual loss exceeded predicted VaR or not. Let the confidence level increase to $\alpha = 99\%$. We take last 500 returns in order to have enough observations below the one percent quantile (due to simulation noise). It is at a cost of assumption, that the stock will behave similarly as in the past two years. The value we get represents VaR, which means that there is a one percent chance that the loss of the portfolio will be greater within a target horizon of one day.

Next, we estimate the parameters needed for the distribution fitting to a normal distribution. There is a formula (equation 2.6) described in section 2.6 which can be applied to a multi-asset portfolio. This method assumes that stock market returns follow a multivariate normal distribution, so we only estimate the covariance matrix of the returns. Applying the formula, we get the pVaR estimates for desired confidence levels. For now we have computed two basic methods, both of them are easy to apply and frequently used.

Last method, evaluated in our model, is a simulation-based Monte Carlo technique. Its application is more complex, time consuming and desires more computer memory than the other two. It is remarkable that even for our simple two asset portfolio it takes so long to compute. The first step is to fit marginal return distributions to the Student's distribution. The degrees of freedom are estimated by the Hill estimator, equation 2.9. Next we run 100,000 simulations. The more repetitions we make, the smaller estimation error gets and the smoother the final curve is. Then, in order to obtain future return distribution of a portfolio, we transform marginal distributions of individual assets into a joint distribution. Having the future distribution of portfolio returns, we can compute its empirical quantile, which represents VaR estimate (multiplied by minus the value of the portfolio). It may be also useful to describe the changes in the estimate of the degrees of freedom. According to Hill, the estimate of the degrees of freedom fluctuated from 3 up to 40 (approximately). Obviously, periods of financial instability are associated with low degrees of freedom and the other way around.

At the end of the model, we compare results of all three approaches. This paragraph briefly described the idea of the empirical analysis and now we move to the interpretation of its results.

Results

First, we look at the VaR estimates at 99% confidence level. The log returns of the portfolio fluctuate between -0.0715 and 0.1088, in other words, the worst loss of a portfolio during the period observed was approximately 6.9% and correspondingly, the biggest gain was around 11.5% with the definition of returns as in equation 2.2. Log returns and estimated VaRs are shown in figure 3.8. The red line represents the pVaR estimates, the green line represents historical VaR estimates and the black, bottom line illustrates simulation based VaR estimates. All methods were evaluated based on past 500 returns. During the analyzed period, we expected to record 53 exceedances; it is a one percent quantile out of 5302 daily observations. In fact, returns exceeded pVaR 194 times, hVaR 73 times and simulation based VaR 53 times, therefore it is the only method which approaches the given confidence level. Parametric VaR estimate exceeded returns in 3.7% cases.

Next, we look at the performance of VaR estimates at 95% confidence level. This time, we expect 278 exceedances over the period of 5552 daily returns. In reality, returns exceeded predicted pVaR 458, historical VaR 309 times and simulation based VaR 221 times. We can notice the same trends in the behaviour of individual VaR measures at both levels of confidence. Here, the VaR exceedance rate was around 8.2% for the pVaR, 5.6% for the historical approach and 3.9% for the Monte Carlo measure.

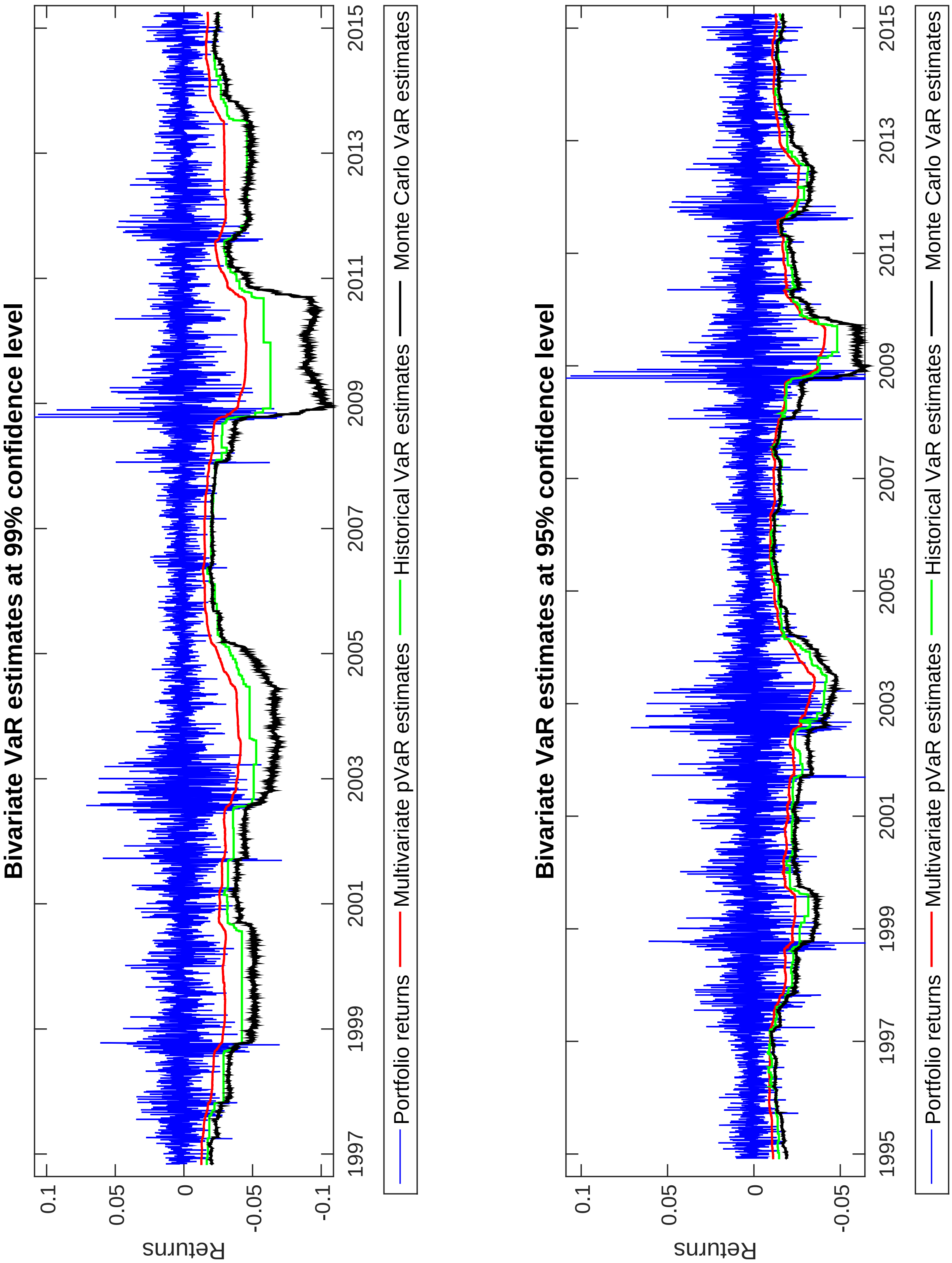


Figure 3.8: Bivariate VaR at various confidence levels.

Discussion

Considering the results of the empirical analysis, it seems that the outcomes of different measuring approaches are straightforward. To model the difference between estimates, we can use an imaginary portfolio. Imagine that on 15th March 2005 a manager invests $1,000,000 in a portfolio which consists of S&P 500 and DAX indices, given that the amount of money invested in each asset is equivalent. VaR estimates for the next day using all three approaches at a confidence level $\alpha = 95\%$ would be following: using parametric approach, VaR is $10,600, hVaR is $11,800 and simulation based VaR equals to $13,700. The difference is even greater considering the 99% confidence level. The pVaR equals to $19,700, hVaR equals to $27,600 and simulation-based VaR is $29,000. This is VaR estimation during the time when volatility of stock returns is lower compared to other periods. As a second example we will demonstrate the same situation in a period of financial crisis, which signifies higher volatility.

Let's move to the 1st January 2009, a period of financial instability. 1-day VaR estimates for the same portfolio as in previous case at a 95% level of confidence are: $36,800 assuming parametric approach, $38,100 assuming historical approach and $62,000 using simulation method. Values corresponding to a 99% confidence level are as follows: $39,600, $63,000 and $100,700. The difference between individual measurement approaches is significant. Regarding preceding examples, we can see that VaR estimates may differ significantly especially in periods of high volatility. Monte Carlo estimate is very sensitive to periods of high volatility, because it estimates the number of degrees of freedom based on the left tail index.

The aim of this thesis is to compare parametric VaR with other methods. It is a frequently used method to compute VaR; however, it leads to much higher number of exceedances than expected. This is due to the heavy-tailed distribution of returns, which is ignored. Parametric estimation produces much more exceedances than other two approaches; it seems that even historical VaR gives more accurate results. We should be careful with this statement, because hVaR depends on historical data set that can well represent the future distribution of major stock market indices, but it might not be well representation in case of more complex assets, such as options. The analysis showed that Monte Carlo simulation is the most conservative of all compared methods, but at a cost of much higher values. It is the only method whose number of exceedances stayed below the expected level and therefore it is the only method which operates at a confidence level we set at the beginning. On the other hand it is not possible to say that this approach is the most accurate, because exceedance rate is not the only measure which matters.

3.3 Final discussion

In the empirical part, we examined the performance of the pVaR measure compared to other common measurement methods. Firstly, we looked at Student's t-distribution which could potentially replace normal distribution in estimating VaR, because returns of an asset are usually not normally distributed. Moreover, we operate on a tail of the distribution, so a little change in the tail of a distribution can lead to significant changes.

Secondly, we compute 1 day VaR using different methods on a short time horizon at different confidence levels, to examine their behaviour in the real world of stock market. The results that we expect are still not straightforward, because the method assuming returns to have Student's t-distribution uses a formula which only confirms the hypotheses for a 99% confidence level. In case of a 95% confidence level, the correction factor drives the value below the pVaR so it has more exceedances.

Section three is devoted to the modelling of multivariate distributions that are not normally distributed. Then we compare the empirical quantile of generated distributions with parametric VaR assuming normal distribution. The results are similar as in the univariate empirical part. For a 95% confidence level, the distribution assuming normal distribution of returns is more conservative than the empirical quantile. Explanation for that is simple. Linear combination of two marginal t-distributed variables does not have t-distribution, it has fat tails, but also, it is more concentrated around 0 than multivariate normal distribution. Cumulative distribution functions of listed distributions cross somewhere between 1st and 5th quantile, so the 5th quantile of multivariate normal distribution is lower than the empirical one. Despite all concerns, the hypotheses that by fitting Student's t-distribution, we get to a conservative value, can still be valid, because the transformation, applied in order to obtain distribution of a portfolio, is not linear.

Finally, previous ideas and findings are adopted in order to compute VaR for two asset portfolio. The pVaR has much more exceedances than we expect. The actual confidence level is much lower than assumed. In this last part, the simulation-based VaR assuming, that returns of individual assets follow

Student's t-distribution is more conservative than parametric VaR approach. The hVaR lies in between, but it is highly dependent on past returns which predict future returns. It can be useful for the assets we chose for this analysis, but the accuracy is questionable when we have more assets in the portfolio. As was already said, more conservative approach is not always more accurate. Measuring the accuracy requires deeper analysis which is beyond the scope of the thesis and it might be a topic for further research. To sum up briefly, we showed that the violation of parametric VaR assumptions leads to remarkable inaccuracy of the measure. Furthermore, we showed that considering a fat-tailed distribution significantly changes the VaR and therefore Monte Carlo simulation performs lower exceedance rate.

Chapter 4

Conclusion

The thesis is an empirical study of VaR risk measure. The main interest was a parametric VaR approach and the intention of the thesis was to evaluate the performance of this approach using a long time series. To test the performance of pVaR, we look at the exceedance rate to see whether it is satisfied given the confidence level. Moreover, we also use different approaches to compare their exceedance rates with the exceedance rate of the pVaR. The theory proposes to estimate future return distribution by Student's t-distribution instead of the normal in order to capture the tails of that distribution better and therefore get more accurate estimates (VaR-x). This method was applied in the empirical part and used as a counterpart to a pVaR. We assumed to obtain lower exceedance rate at a cost of more conservative estimates.

The empirical study was divided into two parts, a univariate and a bivariate analysis. The accuracy of all methods was tested on simulated data as well as on a historical data set. We observed, which estimates we would obtain for a portfolio consisting of stock market indices using parametric, historical and simulation-based (with Student's t-distribution) methods. Univariate analysis has shown that the actual exceedance rate of pVaR is larger than the confidence level set before calculations. The results for the VaR-x are not as straightforward, because the performance of these estimates differs according to the confidence level. The results correspond with the findings of relevant studies. The performance of the hVaR estimates depends on particular stock index, but generally, we can say that it is slightly more conservative than VaR-x. The strength of the VaR-x estimate was in tails of the return distribution. Therefore, we obtained more conservative estimates and lower exceedance rate compared to the pVaR at a 99% confidence level. Meanwhile, for a 95% confidence level, VaR-x estimates have slightly greater exceedance rate than pVaR.

Regarding the bivariate study, the intuition was similar. We only used a Monte Carlo simulation to model the VaR assuming Student's t-distribution instead of the analytic formula used in the univariate case. Simulations predicted the same scenario, that is to have more conservative estimate in case of a 99% confidence level and slightly less conservative in case of a 95% confidence level. Empirical study, however, showed that the Monte Carlo simulation generates more conservative results at both confidence levels. It was also shown, that this method is the only one among the three we worked with, which satisfies the exceedance rate in a long time horizon. Even though the hVaR method overcame the expected exceedance rate, its results were also much better than pVaR (exceedance rate). We also noticed that the Monte Carlo measure was much more conservative than pVaR and hVaR after the periods of greater volatility. This supports the idea, that VaR measures the market risk under normal conditions better.

It is not easy to get acquainted with the amount of literature, which has been written on the theory of VaR. The thesis summarized relevant literature on the implementation of Student's t-distribution to the VaR models. The methods, which were described in the theoretical part of this work, were applied on real data. Moreover, the performance of the techniques was demonstrated by simulations. The thesis does not provide an empirical evidence for saying which approach is more accurate, which could be the topic for further research.

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